HOMOMORPHISMS FROM S^3 TO COMPACT LIE GROUPS UP TO HOMOTOPY

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Let X and Y be Hopf spaces and $\varphi : X \to Y$ be a map. We say φ is a homomorphism up to homotopy, or φ is homomorphic up to homotopy if $\varphi \circ \mu_X$ is homotopic to $\mu_Y \circ (\varphi \times \varphi)$ where μ_X and μ_Y are Hopf space structure maps on X and Y respectively. In this note we consider the following problem: Given a multiplication μ on S^3 and a compact Lie group G and a map $\alpha : S^3 \to G$, when is α homomorphic up to homotopy? Then for our purpose, we propose to construct an obstruction theory. We formulate some properties of these obstructions and by means of this, we investigate the obstructions for each compact connected Lie group.

Key words: Hopf space/Lie group/obstruction/Samelson product

1. Introduction

It is well known that the 3-sphere S^3 has distinct 12 Hopf space structures. Let $\mu : S^3 \times S^3 \to S^3$ be a multiplication on S^3 and G be a compact connected Lie group. In this paper we consider the following problem:

Given a map $\alpha : S^3 \to G$, when is α homomorphic up to homotopy with respect to the given μ?
Then there exists an obseruction $O_\mu(\alpha)$ $(\epsilon \pi_6(G))$ determined by $\alpha \epsilon \pi_3(G)$ and a multiplication μ on S^3 such that α is homomorphic up to homotopy with respect to the given Hopf space structure μ if and only if $O_\mu(\alpha)=0$. We prove:

THEOREM. (1) additive formula; For α, $\beta \epsilon \pi_3(G)$, $O_\mu(\alpha+\beta)=O_\mu(\alpha)+O_\mu(\beta)- \langle \alpha, \beta \rangle$ where $\langle\ ,\ \rangle$ denotes the Samelson product on $\pi_*(G)$.
(2) naturality; For $\alpha \epsilon \pi_3(G)$ and a map $\varphi : G \to H$ where G and H are compact connected Lie groups and φ is a homomorphism up to homotopy, $\varphi_* O_\mu(\alpha)=O_\mu(\varphi_*\alpha)$.
(3) product formula; For $\alpha : S^3 \to G\times H$, $O_\mu(\alpha)=O_\mu(p_1 \circ \alpha)+O_\mu(p_2 \circ \alpha)$ where p_1 and p_2 are projections from $G\times H$ onto each factor.
(4) a relation between the obstructions for multiplications on S^3; Let μ_1 and μ_2 be multiplications on S^3 and α be an element of $\pi_3(G)$. Then $O_{\mu_1}(\alpha)-O_{\mu_2}(\alpha)=\alpha_* d(\mu_1, \mu_2)$ where $d(\mu_1, \mu_2)$ is a H-deviation for μ_1 and μ_2.

COROLLARY 1. Given $\alpha \epsilon \pi_3(G)$ and assume that $\alpha_* : \pi_6(S^3) \to \pi_6(G)$ is surjective (this hypothesis is true on the case that G is a compact connected, simply connected, simple Lie group). Then there exists a Hopf space structure map μ on S^3 such that $O_\mu(\alpha) = 0$, i.e. α is homomorphic up to homotopy with respect to the μ.

COROLLARY 2. Given $\alpha \epsilon \pi_3(G)$ which is divisible by 24, then α is necessarily homomorphic up to homotopy for any Hopf space structure map on S^3.

In this note first we give the definition of an obstruction for the problem and secondly we prove the theorem and at last we investigate obstructions for each compact connected Lie group.

We work in the homotopy category of path connected based spaces of the homotopy type of CW-complexes and do not distinguish between maps and homotopy classes of maps. The notations Z/k is used for the integers mod k and see [M] for basic facts of Hopf spaces and also see [TM] for Lie groups.

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2. Obstructions

Let $\alpha\in\pi_3(G)$. Since S^3 is simply connected, the unique lifting of α from S^3 to the universal covering group $\tilde{G}$ of G (base point preserving map) gives the isomorphism $\pi_*(G)=\pi_*(\tilde{G})$ ($*\geqq 2$). So we suppose that G is a compact connected, simply connected Lie group in this paper.

Let μ be a multiplication on S^3 and α be an element of $\pi_3(G)$. Define the obstruction $O_\mu(\alpha)$ by the homotopy class of the map,

$$\alpha(\mu(x, y))\alpha(y)^{-1}\alpha(x)^{-1} : S^6=S^3\wedge S^3\to G$$

or the deviation between $\alpha\circ\mu$ and $\alpha_1\cdot\alpha_2$ where $\alpha_i=\alpha\circ p_i$ for the projection p_i on each factor.

By using the Puppe exact sequence induced from a cofibre sequence $S^3\vee S^3\to S^3\times S^3\to S^3\wedge S^3$,

$$\begin{array}{ccccccccc} 0 \to & [S^3\wedge S^3, G] & \to & [S^3\times S^3, G] & & \to & [S^3\vee S^3, G] & \to 0, \\ & \Psi & & \Psi & \Psi & & \Psi & \\ & O_\mu(\alpha) & & \alpha\circ\mu & \alpha_1\cdot\alpha_2 & & \alpha\vee\alpha & \end{array}$$

we can know that $O_\mu(\alpha)=0$ is equivalent to α being homomorphic with respect to μ.

Let ψ be the attaching map of 6-cell in the canonical CW-decomposition of $S^3\times S^3$; i.e.

$$\begin{array}{ccc} V^6 & \xrightarrow{\psi} & S^3\times S^3 \\ \uparrow & & \uparrow \\ S^5 & \xrightarrow[{[\iota_1,\ \iota_2]}]{} & S^3\vee S^3. \end{array}$$

Then $d(\alpha\circ\mu,\ \alpha_1\cdot\alpha_2)$ represents the homotopy class of the map $S^6=V^6_+\cup_{S^5}V^6_-\to G$ such that $d(\alpha\circ\mu,\ \alpha_1\cdot\alpha_2)\mid V^6_+=\alpha\circ\mu\circ\psi$ and $d(\alpha\circ\mu,\ \alpha_1\cdot\alpha_2)\mid V^6_-=(\alpha_1\cdot\alpha_2)\circ -\psi$ ($-$ means the reversed orientation). And also the Samelson product $\langle\alpha, \beta\rangle$ is defined by $d(\alpha\cdot\beta, \beta\cdot\alpha)$ for $\alpha, \beta\in\pi_*(G)$.

3. Proof of theorem

Let cat X be the Lusternik-Schnirelmann category of a based space X. This is defined as follows;

It is called cat $X<n$ if and only if n-fold diagonal $\Delta:(X, *)\to(X^n, A)$ is compressible, where $A=\{(x_1,\dots x_n)\in X^n\mid x_i=*$ (base point of X at least one i$\}$, i.e. for the inclusion $j:A\to X^n$, there exists a map $\Delta':X\to A$ such that $\Delta\simeq j\circ\Delta'$ and then cat $X=\min\{n\mid \text{cat } X<n+1\}$ or ∞.

Let $\mathfrak{G}$ be a group and x, y be two elements of $\mathfrak{G}$ and $\mathfrak{H}, \mathfrak{K}$ be subsets of $\mathfrak{G}$. Then a commutator $[x, y]$ is defined by $[x, y]=xyx^{-1}y^{-1}$ and we denote by $[\mathfrak{H}, \mathfrak{K}]$ the subgroup of $\mathfrak{G}$ generated by commutators $[x, y]$ for $x\in\mathfrak{H}$, $y\in\mathfrak{K}$. Let $\mathfrak{G}_0=\mathfrak{G}$ and we put $\mathfrak{G}_i=[\mathfrak{G}, \mathfrak{G}_{i-1}]$ inductively. Then it is called that $\mathfrak{G}$ is nilpotent of class n if $\mathfrak{G}_i$ is trivial for some i and n is the least number among such i's. Let $c:S^3\times S^3\to S^6=S^3\times S^3/S^3\vee S^3$ be the collapsing map and $c(x, y)=x\wedge y$.

LEMMA 1. Any iterated commutators $[[\alpha, \beta], \gamma]$ for $\alpha, \beta, \gamma\in[S^3\times S^3, G]$ are trivial.

PROOF. This follows from that $[S^3\times S^3, G]$ is nilpotent of class less than 3 because of cat $S^3\times S^3=2<3$ (apply [M] the theorem 2.13 p. 287).

LEMMA 2. If $\pi_6(G)=\langle\pi_3(G), \pi_3(G)\rangle$, we have $[c^*\pi_6(G),\ p^*_i\pi_3(G)]=0$ in $[S^3\times S^3, G]$ where p_i ($i=1, 2$) are projections from $S^3\times S^3$ to each i-th factor.

PROOF. By the assumption we may regard an element $\xi\in\pi_6(G)$ as $\langle\alpha, \beta\rangle$ for some $\alpha, \beta\in\pi_3(G)$. So we have $c^*\xi=\xi\circ c=[\alpha\circ p_1,\ \beta\circ p_2]$ in $[S^3\times S^3, G]$. Then by lemma 1, $[c^*\xi,\ p_i^*\gamma]=[[\alpha\circ p_1,\ \beta\circ p_2],\ \gamma\circ p_i]=0$ for $\gamma\in\pi_3(G)$.

Under the same hypothesis of lemma 2, we have;

LEMMA 3. For $\alpha\in\pi_3(G)$ and $\xi\in\pi_6(G)$, the map $(\alpha\circ p_i)\cdot\xi\cdot(\alpha\circ p_i)^{-1}$ ($\in\pi_6(G)$) defined by

$$(\alpha\circ p_i)\cdot\xi\cdot(\alpha\circ p_i)^{-1}(x\wedge y)=(\alpha\circ p_i)(x,\ y)\cdot\xi(x\wedge y)\cdot(\alpha\circ p_i)(x, y)^{-1}$$

for $(x, y)\in S^3\times S^3$ is homotopic to ξ.

PROOF. The elements ξ and $(\alpha\circ p_i)\cdot\xi\cdot(\alpha\circ p_i)^{-1}$ are in $\pi_6(G)$ and the map $c^*:\pi_6(G)\to[S^3\times S^3, G]$ is monomorphic and also the commutator of $c^*\xi$ and $\alpha\circ p_i$, $[c^*\xi,\ \alpha\circ p_i]$ is trivial by lemma 2. These facts deduce the proof.

LEMMA 4. For any compact connected Lie groups, $\pi_6(G)= \langle \pi_3(G), \pi_3(G)\rangle$.
This lemma is proved in the section 4.

Now we proceed to the proof of theorem;

(1) $$O_\mu(\alpha+\beta)(x\wedge y)=\alpha(\mu(x,y))\cdot\beta(\mu(x,y))\cdot\{\alpha(x)\beta(x)\alpha(y)\beta(y)\}^{-1}$$
$$=\alpha(\mu(x,y))\alpha(y)^{-1}\alpha(x)^{-1}\cdot\alpha(x)\alpha(y)\{\beta(\mu(x,y))\beta(y)^{-1}\beta(x)^{-1}\}\alpha(y)^{-1}$$
$$\alpha(x)^{-1}\cdot\alpha(x)\{\alpha(y)\beta(x)\alpha(y)^{-1}\beta(x)^{-1}\}\ \alpha(x)^{-1}.$$

By lemma 3, $\alpha(y)O_\mu(\beta)(x\wedge y)\alpha(y)^{-1}\simeq O_\mu(\beta)(x\wedge y)$, $\alpha(x)O_\mu(\beta)(x\wedge y)\alpha(x)^{-1}\simeq O_\mu(\beta)(x\wedge y)$ and $\alpha(x)\ \langle\alpha,\beta\rangle\ (y\wedge x)\alpha(x)^{-1}\simeq\langle\alpha,\beta\rangle\ (y\wedge x)$. On the other hand from the following commutative diagram,

$$\begin{tikzcd} S^3\times S^3 \arrow[r, "\tau"] \arrow[d, "c"'] & S^3\times S^3 \arrow[d, "c"] \\ S^6 \arrow[r, "-\iota_6"] & S^6 \end{tikzcd}$$

where $\tau(x, y)=(y, x)$, we have $\langle\alpha,\beta\rangle(y\wedge x)\simeq\langle\alpha,\beta\rangle(x\wedge y)\circ(-\iota_6)\simeq-\ \langle\alpha,\beta\rangle\ (x\wedge y)$.
Hence $O_\mu(\alpha+\beta)=O_\mu(\alpha)+O_\mu(\beta)-\langle\alpha,\beta\rangle$.

(2) This is obvious from the definition.

(3) This follows from the canonical decomposition of homotopy groups of the product of spaces; $\pi_*(G\times H)\simeq\pi_*(G)\oplus\pi_*(H)$.

(4) Let μ_1 and μ_2 be multiplicatons on S^3. Since $d(\alpha\circ\mu_1,\ \alpha_1\cdot\alpha_2)=O_{\mu_1}(\alpha)$ and $d(\alpha\circ\mu_2,\ \alpha_1\cdot\alpha_2)=O_{\mu_2}(\alpha)$ by the definition, we have $O_{\mu_1}(\alpha)-O_{\mu_2}(\alpha)=d(\alpha\circ\mu_1,\ \alpha_1\cdot\alpha_2)-d(\alpha\circ\mu_2,\ \alpha_1\cdot\alpha_2)=d(\alpha\circ\mu_1,\ \alpha\circ\mu_2)=\alpha_*d(\mu_1,\mu_2)$.

4. Calculations

We know from the classification theory of compact connected Lie groups that compact connected Lie group G is isomorphic to $T\times G_1\times G_2\times\cdots\times G_k/K$ where T is a torus and G_i ($i=1, 2, \ldots, K$) are compact connected, simply connected, simple Lie groups and K is a finite group contained in the center of $T\times G_1\times G_2\times\cdots\times G_k$. Since $\pi_*(T)=0$ ($*\geqq 2$), by theorem (3) it is sufficient for our purpose to investigate for the cases G is a compact connected, simply connected, simple Lie group. Also we know that compact connected, simply connected, simple Lie groups are classical groups Spin (n), SU(n), Sp(n) and exceptional groups G_2, F_4, E_6, E_7 and E_8.

We know that there exists a following inclusion sequence of Lie groups;

$$S^3=Spin(3)=SU(2)\subset SU(3)\subset G_2\subset Spin(9)\subset F_4\subset E_6\subset E_7\subset E_8,$$

and the following facts about their homotopy groups;

(*) all of the third homotopy groups π_3 of these groups are isomorphic to Z (infinite cyclic group), and isomorphsms are given by inclusions,

and also inclusions give epimorphisms;

(**) $\pi_6(S^3)\cong Z/12\to\pi_6(SU(3))\cong Z/6\to\pi_6(G_2)\cong Z/3\to\pi_6(Spin(9))\cong\pi_6(F_4)\cong\pi_6(E_6)\cong\pi_6(E_7)\cong\pi_6(E_8)\cong 0$,

and

(***) $\pi_6(S^3)$ is generated by $\langle\iota_3,\iota_3\rangle$.

Thus we investigate the following cases;

(1) Spin(n)

n=3; Spin(3)=S^3=quarternions of norm 1. $\pi_3(S^3)=Z[\iota_3]$ and $\pi_6(S^3)=Z/12[\omega]$ where ω is the "Blakers-Massey" element $d(q,\ q\circ\tau)=\langle\iota_3,\iota_3\rangle$ where q is the quarternionic multiplication on S^3 and τ is a coordinate changing map $\tau(x, y)=(y, x)$. A multiplication μ on S^3 is determined by a number $k(\mu)$ mod 12 such that $O_\mu(\iota_3)=k(\mu)\omega$.

Then by theorem (1) and the induction, we have

PROPOSITION 1. $O_\mu(\lambda\iota_3)=\lambda O_\mu(\iota_3)-\{\lambda(\lambda-1)/2\}\omega=[\lambda k(\mu)-\{\lambda(\lambda-1)/2\}]\omega$.

Specially,

COROLLARY 3. [Arkowitz-Curjel [1], McGibbon [3]] If $k(\mu)=0$ (when $\mu=q$), the power map $\lambda\iota_3$ is homomorphic up to homotopy if and only if $\lambda(\lambda-1)\equiv 0 \bmod 24$.

COROLLARY 4. If $\lambda\equiv 0 \bmod 24$, for any multiplication μ on S^3, $O_\mu(\lambda\iota_3)=0$ i.e. $\lambda\iota_3$ is homomorphic up to homotopy.

$n=4$; We know the following diagram;

$$\begin{tikzcd}
\mathrm{Spin}(3) \arrow[d, equal] & \mathrm{Spin}(4) \arrow[d, equal] & \\
S^3 \arrow[r, "\Delta"] \arrow[d, "\rho"'] & S^3\times S^3 \arrow[r, shift left, "\tilde{q}"] \arrow[d, "\pi"'] & S^3 \arrow[l, shift left, "\tilde{\sigma}"] \arrow[d, equal] \\
SO(3) \arrow[r, "i"] & SO(4) \arrow[r, shift left, "p"] & S^3 \arrow[l, shift left, "\sigma"]
\end{tikzcd}$$

where Δ is a diagonal, p is the principal SO(3) bundle projection and σ is a section $S^3\to SO(4)$ defined by $\sigma(x)y=xy$ and $\tilde{q}(x, y)=x^{-1}y$ for x, y $\in S^3=$quarternions and $\tilde{\sigma}$ is a lifting of σ and ρ, π are universal covering maps. So this case is reduced to the case $n=3$ by theorem (3).

$n\geqq 5$; $\pi_3(\mathrm{Spin}(n))\doteqdot Z$ which is generated by the image of $\tilde{\sigma}:S^3\to \mathrm{Spin}(n)$ by the inclusion map and $\pi_6(\mathrm{Spin}(n))=0$. In this case the obstruction class $O_\mu(\ \ )$ is always trivial.

(2) SU(n)

$n=2$; $SU(2)=S^3$

$n=3$; $\pi_3(SU(3))=Z[\iota]$ where ι is the image of ι_3 by the inclusion $SU(2)\to SU(3)$ and $\pi_6(SU(3))=Z/6[v]$ where v is the image of $\langle\iota_3, \iota_3\rangle$ by the inclusion. In this case we have the analogous results of proposition 1 and corollary 3 and 4 except for the order of $\pi_6(SU(3))$.

$n\geqq 4$; $\pi_3(SU(n))\cong Z$ which is generated by the image of ι_3 and $\pi_6(SU(n))=0$. Hence the obstruction is always trivial.

(3) Sp(n)

$n=1$; $Sp(1)=S^3$

$n\geqq 2$; $\pi_3(Sp(n))\cong Z$ which is generated by the image of ι_3 and $\pi_6(Sp(n))\cong 0$. In this case the obstruction is always trivial too.

(4) G_2

$\pi_3(G_2)\cong Z$ which is generated by the image of ι_3 and $\pi_6(G_2)=Z/3[\xi]$ where ξ is the image of $\langle\iota_3, \iota_3\rangle$ by the inclusion. In this case we have an alalogous results of proposition 1 and corollary 3 and 4 except for the order of $\pi_6(G_2)$.

(5) F_4, E_6, E_7 and E_8

$$\pi_3(F_4)\cong\pi_3(E_6)\cong\pi_3(E_7)\cong\pi_3(E_8)\cong Z$$

which are generated by the image of ι_3 by the inclusions and π_6 of these groups are trivial. Hence the obstructions are always trivial.

Thus we have investigated all of the cases that G is any compact connected, simply connected, simple Lie group. At last we give;

PROOF of corollary 1: Any elements of $\pi_6(S^3)$ are of the type $d(\mu_1, \mu_2)$. Given μ such that $O_\mu(\iota_3)=k(\mu)\omega$, we can take μ_0 such that $-\alpha_* d(\mu_0, \mu)=O_\mu(\alpha)$. Then by theorem (4), $O_{\mu_0}(\alpha)=O_\mu(\alpha)+\alpha_* d(\mu_0, \mu)=0$.

PROOF of corollary 2: This follows from the classification of compact connected Lie groups and corollary 4.

PROOF of lemma 4: This follows from the classification of compact connected Lie groups and (* *) and (* * *).

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ERRATUM

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The correction is to the assertion in the parenthesis of Corollary 1 on page 1 of my paper.

It appears that the assertion is wrong and should make good a deficiency of conditions as follows;

(this hypothesis is true on the case that G is a compact connected, simply connected, simple Lie group and α is not divisible by 2 nor 3).